AXISYMMETRIC P-HARMONIC MODES IN CIRCULAR CONES: NODAL CLASSIFICATION AND TWO-TERM ASYMPTOTICS

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ABSTRACT. We study homogeneous axisymmetric solutions of the p-Laplace equation in a circular cone with zero lateral boundary values. A lifted phase gives one positive and one negative homogeneity exponent for each nodal index. An energy argument excludes an additional continuously differentiable profile vanishing at the axis. We obtain two-term high-index asymptotics, including an axis-layer phase shift given by a convergent improper integral. The two branches have opposite leading terms and an explicit limiting sum. The analysis concerns the axisymmetric class and does not provide an expansion of an arbitrary solution at a conical point.

1. INTRODUCTION AND MAIN RESULTS

Homogeneous solutions are basic model functions for elliptic equations near conical boundary points. For the p-Laplace equation, separation of the radial variable leads to a nonlinear angular equation containing the homogeneity parameter both in its principal part and in its zero-order term. Its nodal solutions are therefore not the eigenfunctions of a fixed linear angular operator, and ordinary superposition is unavailable.

Król constructed separable solutions and studied their behavior at boundary singularities and narrow cones [5, 6]. The planar regular and singular branches are discussed in [4]. Existence and uniqueness of the positive separable branches in general smooth cones are treated by Porretta and Véron [7], using a different approach to earlier results of Tolksdorf. Borghol and Véron [1, Section 4] construct signed separable solutions in higher dimensions by lifting planar profiles. Those constructions show that the mere existence of infinitely many signed separable solutions is not a novelty claim.

A separate manuscript of the authors [3] concerns only the first positive branch and the limit in which the cone opening tends to zero. It derives a convergent Laurent expansion in that opening. The present paper instead fixes the opening, lets the nodal index tend to infinity, and treats both signs of the homogeneity exponent. Thus the two asymptotic regimes and their main theorems are complementary; neither result contains the other.

Here the angular domain is a circular cap of arbitrary fixed opening. We give a self-contained treatment of its axisymmetric nodal branches and retain the constant terms in their high-index asymptotics. The singular endpoint at the axis contributes a phase shift that is lost by averaging only on intervals separated from that endpoint. The main point of the asymptotic argument is a uniform bound on the tail of the rescaled endpoint integral.

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Fix an integer $n \geq 3$, a real number $p > 1$, and $0 < \alpha < \pi$. Write $r = |x|$ and $\theta = \arccos(x_n/|x|)$. The circular cone is

$$\mathcal{C}_\alpha = \{x \in \mathbb{R}^n \setminus \{0\} : 0 \leq \theta < \alpha\}.$$

We seek nonzero real functions

$$u(r,\theta) = r^\kappa \phi(\theta), \qquad \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad \phi'(0) = 0, \quad \phi(\alpha) = 0. \tag{1.1}$$

Throughout, a regular profile means $\phi \in C^1([0,\alpha])$ satisfying the angular equation weakly, with the indicated axial condition. No regularity of u at the vertex is required; negative homogeneities are allowed.

Set

$$\kappa_*^+ = \max\left(0, \frac{p-n}{p-1}\right), \qquad \kappa_*^- = \min\left(0, \frac{p-n}{p-1}\right). \tag{1.2}$$

Theorem 1.1 (Nodal branches). *For each integer $j \geq 0$, there is exactly one exponent $\kappa_j^+ > \kappa_*^+$ and exactly one exponent $\kappa_j^- < \kappa_*^-$ for which (1.1) has a regular profile with exactly j zeros in $(0,\alpha)$. The profile is unique after normalization $\phi(0) = 1$. Every interior zero is simple, and*

$$\kappa_0^+ < \kappa_1^+ < \cdots \longrightarrow +\infty, \qquad \kappa_0^- > \kappa_1^- > \cdots \longrightarrow -\infty.$$

These two sequences exhaust the exponents of regular nonzero axisymmetric profiles. Each shooting root is transverse.

To state the asymptotics, define

$$D(v) = \cos^2 v + (p-1)\sin^2 v, \qquad \mu_p = \frac{p+2}{4p}. \tag{1.3}$$

Let Θ_0 be the regular solution on $s > 0$ of

$$D(\Theta_0)\Theta_0' = p - 1 - \frac{n-2}{s}\sin\Theta_0\cos\Theta_0, \qquad \Theta_0(s) = \frac{p-1}{n-1}s + O(s^3). \tag{1.4}$$

Theorem 1.2 (Two-term asymptotics). *The improper integral*

$$L_{p,n} = \int_0^\infty \frac{\sin\Theta_0(s)\cos\Theta_0(s)}{s}\,ds \tag{1.5}$$

converges. Its tail beyond S is $O(S^{-1})$ as $S \to \infty$. For each fixed p, n, α as above,

$$\kappa_j^+ = \frac{p\pi}{2(p-1)\alpha}\left(j + \frac{1}{2}\right) - \frac{(n-p)(p+2)}{4p(p-1)} + \frac{(n-2)L_{p,n}}{(p-1)\alpha} + o(1), \tag{1.6}$$

$$\kappa_j^- = -\frac{p\pi}{2(p-1)\alpha}\left(j + \frac{1}{2}\right) - \frac{(n-p)(p+2)}{4p(p-1)} - \frac{(n-2)L_{p,n}}{(p-1)\alpha} + o(1). \tag{1.7}$$

Corollary 1.3. *The consecutive spacings tend to $p\pi/[2(p-1)\alpha]$ on the positive branch and its negative on the negative branch. Moreover,*

$$\lim_{j\to\infty}(\kappa_j^+ + \kappa_j^-) = -\frac{(n-p)(p+2)}{2p(p-1)}. \tag{1.8}$$

The index j counts zeros of a one-dimensional angular profile. It does not count the full multidimensional spectrum or its multiplicities. The $o(1)$ statements keep the cone opening fixed; they do not assert uniformity as $\alpha \to 0$ or $\alpha \to \pi$.

2. The angular equation and the axial endpoint

Put $m = n - 2$ and

$$H = \kappa^2\phi^2 + \phi'^2, \qquad \lambda = \kappa((p-1)\kappa + n - p).$$

Separation in (1.1) gives

$$(\sin^m\theta\, H^{(p-2)/2}\phi')' + \lambda \sin^m\theta\, H^{(p-2)/2}\phi = 0. \tag{2.1}$$

Products of the form $H^{(p-2)/2}\phi$ or $H^{(p-2)/2}\phi'$ at $H = 0$ are interpreted by continuity when $\kappa \neq 0$. Multiplication by ϕ, followed by integration, gives

$$\int_0^\alpha H^{(p-2)/2}\phi'^2 \sin^m\theta\, d\theta = \lambda \int_0^\alpha H^{(p-2)/2}\phi^2 \sin^m\theta\, d\theta. \tag{2.2}$$

The boundary term at the axis vanishes: for $\kappa \neq 0$ it is bounded by a constant times $\sin^m\theta\, H^{p/2}$. The two integrals are finite. A nonzero profile with zero value at α makes the left-hand side strictly positive. Hence $\lambda > 0$, which gives the two intervals in (1.2). When $\kappa = 0$, the angular equation is the spherical p-Laplace equation; testing with ϕ makes ϕ constant, and its boundary value makes it zero. Thus $\kappa = 0$ is excluded without using the expressions with $H^{(p-2)/2}\phi$.

Lemma 2.1 (No hidden axial branch)**.** *If $\kappa \neq 0$ and a regular profile satisfies $\phi(0) = \phi'(0) = 0$, then it vanishes identically.*

Proof. Write $q = \phi'$ and $E = \kappa^2\phi^2 + q^2$. On any component where $E > 0$, equation (2.1) is a smooth ordinary differential equation and can be expanded as

$$q' = -a(\phi, q)\cot\theta\, q + G(\phi, q), \tag{2.3}$$

where

$$a(\phi, q) = m\frac{\kappa^2\phi^2 + q^2}{\kappa^2\phi^2 + (p-1)q^2} \geq 0$$

and

$$G(\phi, q) = -\frac{(p-2)\kappa^2\phi q^2 + \lambda\phi(\kappa^2\phi^2 + q^2)}{\kappa^2\phi^2 + (p-1)q^2}.$$

The denominator is comparable to E, so $|G(\phi, q)| \leq CE^{1/2}$. For $0 < \theta < \pi/2$ this gives

$$E' = 2\kappa^2\phi q + 2qG - 2a\cot\theta\, q^2 \leq CE. \tag{2.4}$$

If a component of $\{E > 0\}$ has left endpoint $b \geq 0$ in this interval, then $E(b) = 0$ by continuity, including $b = 0$. Integrating (2.4) from $\delta > b$ to an interior point and letting $\delta \downarrow b$ gives $E = 0$, a contradiction. Thus E vanishes near the axis.

On an interval separated from the axis, the vector field $(\phi, q) \mapsto (q, q')$ extends continuously by zero at the origin and is locally Lipschitz. Indeed, its rational expressions are homogeneous of degree one with smooth bounded angular derivatives on an ellipse. Uniqueness therefore extends the zero solution through $(0, \alpha]$. □

Remark 2.2. The sign of $\cot\theta$ is only used on a short interval near zero. Lemma 2.1 consequently applies also when $\alpha > \pi/2$. It avoids a classification of possible singular phase limits at the axis.

3. Lifted phase, regular shooting, and all nodal branches

After multiplying by a constant, every nonzero regular profile has $\phi(0) = 1$. Introduce a continuous real phase and a signed amplitude by

$$\kappa\phi = R\cos\Theta, \qquad -\phi' = R\sin\Theta, \qquad R(0) = \kappa, \quad \Theta(0) = 0. \tag{3.1}$$

The pair (ϕ, ϕ') cannot vanish at a positive angle by the local uniqueness used in Lemma 2.1. Direct calculation gives

$$\Theta' = \frac{\kappa(p-1) + (n-p)\cos^2\Theta - m\cot\theta\sin\Theta\cos\Theta}{D(\Theta)} =: F(\theta, \Theta; \kappa) \tag{3.2}$$

and

$$\frac{R'}{R} = \frac{\sin\Theta\{((p-2)\kappa + n - p)\cos\Theta - m\cot\theta\sin\Theta\}}{D(\Theta)}. \tag{3.3}$$

The amplitude equation is regular at every zero of $\cos\Theta$. The amplitude stays nonzero, with the sign of κ.

3.1. **A regular solution at the singular endpoint.**

For completeness, the regular initial condition in (3.2) can be imposed without an unspecified singular initial-value theorem. Set

$$c_\kappa = \frac{\kappa(p-1) + n - p}{n-1}, \qquad \Theta = \theta(c_\kappa + z).$$

Taylor expansion of the analytic functions in (3.2) reduces it, near $(\theta, z) = (0, 0)$, to

$$\theta z' + (n-1)z = \theta^2 H_1(\theta^2, z; \kappa), \tag{3.4}$$

where H_1 is analytic locally, uniformly for κ in a fixed compact set. The corresponding integral equation is

$$z(\theta) = \theta^{-(n-1)} \int_0^\theta t^n H_1(t^2, z(t); \kappa)\, dt. \tag{3.5}$$

On a ball in the norm $\sup_{0<\theta\le\delta} |z(\theta)|/\theta^2$, its right side is bounded independently of small δ, and its Lipschitz constant is $O(\delta^2)$. Choosing the ball first and then δ gives a unique fixed point. Differentiating this contraction with respect to κ gives continuous parameter dependence and the corresponding differentiated equation. In particular,

$$\Theta(\theta; \kappa) = c_\kappa\theta + O(\theta^3). \tag{3.6}$$

The same construction, with the evident coefficient changes, gives the regular solution of (1.4).

Any regular profile normalized by $\phi(0) = 1$ belongs to this shooting branch. To check this, integrate (2.1) from zero. Since $H \to \kappa^2 > 0$, one obtains $\phi' = -\lambda\theta/(n-1) + o(\theta)$. Thus $\Theta/\theta \to c_\kappa$. The bounded solution of (3.4) satisfies (3.5), which also improves this initial estimate to $z = O(\theta^2)$.

Since D is bounded above and bounded away from zero, (3.2) extends to every compact subinterval of $(0, \pi)$. Equation (3.3) then recovers a regular profile across all its zeros.

3.2. **Monotonicity in the homogeneity parameter.** Let $Z = \partial_\kappa \Theta$. Differentiation gives

$$Z' = F_\Theta Z + \frac{p-1}{D(\Theta)}, \qquad Z(\theta) = \frac{p-1}{n-1}\theta + O(\theta^3). \tag{3.7}$$

On an interval $[\delta, \alpha]$ with sufficiently small $\delta > 0$, variation of constants implies $Z > 0$. Hence

$$\partial_\kappa \Theta(\alpha; \kappa) > 0 \tag{3.8}$$

for every real κ in the phase equation. This argument does not divide by ϕ and remains valid after any number of nodal crossings.

At a nodal level one has the exact identity

$$\Theta = \frac{\pi}{2} + \ell\pi \quad \Longrightarrow \quad \Theta' = \kappa. \tag{3.9}$$

Therefore each such crossing is simple and has the sign of κ. A fixed nodal level cannot be crossed twice in that direction without an intervening crossing in the opposite direction, which (3.9) excludes.

3.3. **Range of the shooting map.** For $\kappa > \kappa_*^+$, the regular phase starts positive and cannot cross zero downwards, since $F(\theta, 0; \kappa) = \kappa(p-1) + n - p > 0$. At the lower endpoint $\kappa = \kappa_*^+$ the final phase is strictly below $\pi/2$. If $p \geq n$, that endpoint has the solution $\Theta \equiv 0$. If $p < n$, the endpoint is $\kappa = 0$; the level $\pi/2$ is then a solution on every interval away from zero. Uniqueness prevents the regular branch starting below it from reaching it at a finite positive angle.

On $[\alpha/2, \alpha]$, the lower-order terms of (3.2) are bounded uniformly in κ. Thus, with positive constants depending on the fixed parameters,

$$\Theta'(\theta; \kappa) \geq c\kappa - C \quad \text{for large positive } \kappa.$$

Since $\Theta(\alpha/2; \kappa) \geq 0$, the final phase tends to $+\infty$ with κ. Strict monotonicity and continuity give exactly one solution of

$$\Theta(\alpha; \kappa_j^+) = \frac{\pi}{2} + j\pi. \tag{3.10}$$

For $\kappa < \kappa_*^-$, the phase stays negative and its final value tends to $-\infty$ as $\kappa \to -\infty$. At the upper endpoint it is strictly above $-\pi/2$: when $p \leq n$ the endpoint solution is zero, and when $p > n$ the barrier at $\kappa = 0$ is $-\pi/2$. Consequently there is exactly one solution of

$$\Theta(\alpha; \kappa_j^-) = -\frac{\pi}{2} - j\pi.$$

Equation (3.9) counts exactly j interior zeros, and (3.8) proves transversality. The sequences cannot have a finite accumulation point because the phase at a fixed angle is continuous and finite there. Together with the energy exclusion and Lemma 2.1, this proves Theorem 1.1.

Corollary 3.1 (Dependence on the opening)**.** *Each branch is continuously differentiable in* $\alpha \in (0, \pi)$*, and*

$$\frac{d\kappa_j^\pm}{d\alpha} = -\frac{\kappa_j^\pm}{\partial_\kappa \Theta(\alpha; \kappa_j^\pm)}.$$

Thus positive exponents decrease and negative exponents increase when the cone is widened.

Proof. Apply the implicit function theorem to the respective terminal phase equation, using (3.8) and (3.9). $\square$

4. A phase with constant leading speed

Define the strictly increasing function

$$\mathcal{D}(v)=\int_0^v D(t)\,dt=\frac{p}{2}v+\frac{2-p}{4}\sin 2v, \qquad \Lambda=\mathcal{D}(\pi)=\frac{p\pi}{2}. \tag{4.1}$$

In the variable $Q=\mathcal{D}(\Theta)$, equation (3.2) is

$$Q'=\kappa(p-1)+(n-p)G_1(Q)-m\cot\theta\,G_2(Q), \tag{4.2}$$

where the smooth Λ-periodic functions satisfy

$$G_1(\mathcal{D}(v))=\cos^2 v, \qquad G_2(\mathcal{D}(v))=\sin v\cos v.$$

Their means with respect to Q, rather than to v, are

$$\langle G_1\rangle=\frac{1}{\Lambda}\int_0^\pi \cos^2 v D(v)\,dv=\frac{p+2}{4p}, \qquad \langle G_2\rangle=0. \tag{4.3}$$

Indeed, the required elementary integrals are $\int_0^\pi \cos^4 v\,dv=3\pi/8$ and $\int_0^\pi \sin^2 v\cos^2 v\,dv=\pi/8$; the mixed odd integrals vanish.

Lemma 4.1 (Averaging away from the axis)**.** *Let $0<\delta<\alpha$ and let $a\in C^1([\delta,\alpha])$. For any smooth periodic G,*

$$\int_\delta^\alpha a(\theta)G(Q_\kappa(\theta))\,d\theta=\langle G\rangle\int_\delta^\alpha a(\theta)\,d\theta+o(1) \quad (|\kappa|\to\infty).$$

Proof. Subtract the mean and let P be a bounded periodic primitive of the resulting function. On this fixed interval, $Q'=\kappa(p-1)+O_\delta(1)$ and $Q''=O_\delta(|\kappa|)$. For sufficiently large $|\kappa|$, Q' is nonzero. Integration by parts gives

$$\int_\delta^\alpha a(G(Q)-\langle G\rangle)\,d\theta=\left[\frac{aP(Q)}{Q'}\right]_\delta^\alpha-\int_\delta^\alpha P(Q)\left(\frac{a}{Q'}\right)'\,d\theta.$$

Both terms are $O_\delta(|\kappa|^{-1})$. □

It follows, by bounding the interval $(0,\delta)$ by its length, that

$$\int_0^\alpha \cos^2\Theta(\theta;\kappa)\,d\theta=\alpha\mu_p+o(1). \tag{4.4}$$

The same argument is insufficient for the cotangent term in (4.2); the next section treats that term.

5. The axial layer and its uniform tail

First let $\kappa=K\to+\infty$. For $s=K\theta$, put

$$q_K(s)=\mathcal{D}(\Theta(s/K;K)), \qquad a_K(s)=K^{-1}\cot(s/K).$$

The equation becomes

$$q_K'=p-1+\frac{n-p}{K}G_1(q_K)-ma_KG_2(q_K). \tag{5.1}$$

On each fixed interval $[0,S]$, the regular construction in (3.5) and ordinary continuous dependence away from zero give convergence to $q_0=\mathcal{D}(\Theta_0)$, where

$$q_0'=p-1-\frac{m}{s}G_2(q_0). \tag{5.2}$$

Near zero, $q_K = O(s)$ uniformly for large K, so the singular integrands $a_K G_2(q_K)$ are uniformly bounded there. Thus

$$\lim_{K\to\infty} \int_0^S a_K(s) G_2(q_K(s))\, ds = \int_0^S \frac{G_2(q_0(s))}{s}\, ds. \tag{5.3}$$

Lemma 5.1 (Uniform tail estimate)**.** *Fix $0 < \delta < \min(\alpha, \pi/2)$. For all sufficiently large S, and then all sufficiently large K with $K\delta > S$,*

$$\left| \int_S^{K\delta} a_K(s) G_2(q_K(s))\, ds \right| \le \frac{C}{S} + o_K(1). \tag{5.4}$$

Also $\left| \int_S^T G_2(q_0(s))\, ds/s \right| \le C/S$ for every $T > S$ sufficiently large, and the integral to infinity exists.

Proof. On the specified interval, $0 < a_K(s) \le 1/s$ and a_K is decreasing. Choose S so large that the $ma_K G_2$ term is less than $(p-1)/4$ in absolute value, and then choose K large for the G_1/K term. Equation (5.1) yields

$$0 < c \le q_K' \le C. \tag{5.5}$$

Differentiating (5.1) shows

$$|q_K''| \le C(K^{-1} + |a_K'| + a_K).$$

By (4.3), G_2 has a bounded periodic primitive P. Therefore

$$\int_S^{K\delta} a_K G_2(q_K)\, ds = \left[\frac{a_K P(q_K)}{q_K'} \right]_S^{K\delta} - \int_S^{K\delta} P(q_K) \left(\frac{a_K}{q_K'} \right)' ds.$$

The boundary terms are at most C/S and

$$\left| \left(\frac{a_K}{q_K'} \right)' \right| \le C\big(|a_K'| + a_K^2 + K^{-1} a_K\big).$$

Here the harmless factor $a_K |a_K'|$ was absorbed into $|a_K'|$ by taking $S \ge 1$. Monotonicity and $a_K \le 1/s$ give

$$\int_S^{K\delta} |a_K'|\, ds \le S^{-1}, \qquad \int_S^{K\delta} a_K^2\, ds \le S^{-1}, \qquad K^{-1} \int_S^{K\delta} a_K\, ds = O(K^{-1} \log K).$$

This proves (5.4). Repeating the calculation with $a_K = 1/s$ and $q_K = q_0$ gives an integrable bound C/s^2 for the differentiated factor. The terminal boundary term tends to zero. Hence the improper integral exists and its tail is $O(S^{-1})$. ☐

Combine (5.3), Lemma 5.1, and Lemma 4.1 on $[\delta, \alpha]$. Taking $K \to \infty$ first and then $S \to \infty$ gives

$$\int_0^\alpha \cot\theta \sin\Theta(\theta; K) \cos\Theta(\theta; K)\, d\theta = L_{p,n} + o(1). \tag{5.6}$$

This proves the convergence asserted in (1.5) and identifies the endpoint contribution without an unjustified exchange of infinite-range limits.

For $\kappa = -K$, use $\Xi = -\Theta$ and $q_K = \mathcal{D}(\Xi(s/K; -K))$. Since $\mathcal{D}$ and G_2 are odd and G_1 is even, the equation is

$$q_K' = p - 1 - \frac{n-p}{K} G_1(q_K) - m a_K G_2(q_K).$$

The limiting equation is again (5.2), and all tail bounds are unchanged. The original cotangent integral changes sign. Consequently

$$\int_0^\alpha \cot\theta \sin\Theta(\theta;-K)\cos\Theta(\theta;-K)\,d\theta = -L_{p,n} + o(1). \tag{5.7}$$

6. Quantization and exact checks

At the terminal phase of the positive branch,

$$\mathcal{D}(\Theta(\alpha;\kappa_j^+)) = \frac{p\pi}{4}(2j+1).$$

Integrating (4.2) from zero to α yields the exact quantization identity

$$\begin{aligned}\frac{p\pi}{4}(2j+1) = \kappa_j^+(p-1)\alpha + (n-p)\int_0^\alpha \cos^2\Theta\,d\theta \\ - m\int_0^\alpha \cot\theta\sin\Theta\cos\Theta\,d\theta.\end{aligned} \tag{6.1}$$

Since $\kappa_j^+ \to \infty$, substituting (4.4) and (5.6) proves (1.6). On the negative branch both the terminal phase and the cotangent integral change sign, whereas the mean of G_1 does not. This proves (1.7). Taking consecutive differences and the sum proves Corollary 1.3.

6.1. **The linear case.** When $p = 2$, the limiting equation is generated by the regular radial solution

$$g'' + \frac{n-2}{s}g' + g = 0, \qquad g(s) = c\,s^{-\nu}J_\nu(s), \quad \nu = \frac{n-3}{2},$$

with c chosen so that $g(0) = 1$. The classical asymptotics of the Bessel zeros [2, Section 10.21(vi)] give, for the phase lifted from zero,

$$\Theta_0(s) = s - \frac{n-2}{4}\pi + o(1).$$

Integrating (1.4) in this case gives

$$s - \Theta_0(s) = (n-2)\int_0^s \frac{\sin\Theta_0(t)\cos\Theta_0(t)}{t}\,dt.$$

Hence $L_{2,n} = \pi/4$. Independently, the full linear angular equation depends on κ only through $\kappa(\kappa+n-2)$. The two roots for each angular eigenfunction therefore obey the exact identity

$$\kappa_j^+ + \kappa_j^- = 2 - n.$$

This agrees with (1.8). For $n = 4, p = 2$, a normalized angular profile is proportional to $\sin((\kappa+1)\theta)/\sin\theta$. Thus the positive exponents are exactly $(j+1)\pi/\alpha - 1$, in agreement with both constant contributions in (1.6).

6.2. **The conformal line.** If $p = n$, equation (2.1) is unchanged by $\kappa \mapsto -\kappa$, since $\lambda = (p-1)\kappa^2$. Theorem 1.1 then gives the exact relation

$$\kappa_j^- = -\kappa_j^+,$$

with the same normalized profile. Away from this line and the linear case, the argument proves the limiting relation (1.8), without asserting an exact reflection law.

7. Scope and further questions

The result gives the nodal classification and the first two high-index terms for regular axisymmetric profiles in a circular cap. Its proof uses scalar shooting and a singular-endpoint averaging argument. In particular, it does not require the classification of nonaxisymmetric homogeneous solutions.

Two distinct limiting procedures should be kept separate. Król's narrow-cone problem lets $\alpha \to 0$ for the first positive profile. Here α is fixed and the nodal index tends to infinity. The endpoint equation resembles a narrow-cone scaling, but its complete phase on the half-line, including arbitrarily many zeros, enters (1.5).

No assertion here establishes a nonlinear modal expansion for an arbitrary solution near the vertex, a complete spectrum for arbitrary cone sections, or weighted solvability in the sense of a full nonlinear Kondrat'ev theory. Such statements need additional compactness, rigidity, and all-scale control.

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